Using Exterior Powers in Codimension Two Iwasawa Theory

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Historical Background

In 1959, Kenkichi Iwasawa [1] began the study of the eponymous "Iwasawa Theory"by considering so-called "$\mathbb{Z}_p$-extenions"of number fields, defined to be number fields K whose Galois group is isomorphic to the additive group of p-adic integers $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. While motivated by cyclotomic field theory and arising from the general background of recent advances on the problem of infinite towers of number fields in class field theory, researchers soon realized the theoretical depth possessed by $\mathbb{Z}_p$-extensions. Indeed, concurrent advances in p-adic analysis and the definition of new p-adic L-functions led to Iwasawa's "main conjectures", which essentially claim that the analytic and module-theoretic method of defining these p-adic L-functions are in fact equivalent.

Notation and Definitions

We now provide some essential definitions for the rest of the poster. Let K be an imaginary number field in which a prime p splits, with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ lying over p inside K. Set $G_{\mathbb{Q}}$ and G_K the absolute Galois groups of $\mathbb{Q}$ and K, respectively, and similarly $\mathbb{Q}_{\text{cyc}}$ and K_{cyc} the cyclotomic $\mathbb{Z}_p$ extensions. Let $K(\mathfrak{p}^\infty)_{\mathbb{Z}_p}$ be the unique $\mathbb{Z}_p$-extension of K which is unramified outside p. Set $\Gamma_p, \Gamma_{\text{cyc}}$, and $\tilde{\Gamma}$ the Galois groups $\text{Gal}(K(\mathfrak{p}^\infty)_{\mathbb{Z}_p}/K) \cong \mathbb{Z}_p$, $\text{Gal}(K_{\text{cyc}}/K) \cong \mathbb{Z}_p$, and $\text{Gal}(\tilde{K}_\infty/K) \cong \mathbb{Z}_p^2$, where $\tilde{K}_\infty$ is the compositum of all $\mathbb{Z}_p$-extensions of K. We let E be an elliptic curve over $\mathbb{Q}$ which has good supersingular reduction at p and satisfies $a_p(E) = 0$, and f_E the associated weight two cuspidal newform with p-adic Tate module $T_p(E)$. Define

$$T_{\rho_{4,2}} := T_p(E) \widehat{\otimes}_{\mathbb{Z}_p} \text{Ind}_{G_K}^{G_{\mathbb{Q}}}(\mathbb{Z}_p[[\Gamma_p]](\kappa_p^{-1})) \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma_{\text{cyc}}]](\kappa_{\text{cyc}}^{-1}),$$

where $\widehat{\otimes}_{\mathbb{Z}_p}$ denotes the completed tensor product over $\mathbb{Z}_p$ (see the Stacks project for a definition), $\kappa_{\text{cyc}} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\text{cyc}} \hookrightarrow \text{GL}_1(\mathbb{Z}_p[[\Gamma_{\text{cyc}}]])$ is the tautological character, and $\mathbb{Z}_p[[\Gamma_{\text{cyc}}]](\kappa_{\text{cyc}}^{-1})$ is the Iwasawa algebra over Γ_{cyc} endowed with the natural action from κ_{cyc}. The module $T_{\rho_{4,2}}$ is free of rank 4, and induces a Galois representation $\rho_{4,2} : G_{\mathbb{Q}} \to \text{GL}_4(\mathbb{Z}_p[[\tilde{\Gamma}]])$. We further abstract to define a discrete $\mathbb{Z}_p[[\tilde{\Gamma}]]-$module

$$D_{\rho_{4,2}} := T_{\rho_{4,2}} \otimes_{\mathbb{Z}_p[[\tilde{\Gamma}]]} \text{Hom}_{\text{cont}}\left(\mathbb{Z}_p[[\tilde{\Gamma}]], \frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right).$$

With these in place, one may consider define a family of p-adic L-functions (see the work of Haruzo Hida) which we denote $\theta_{4,2}^{\bullet;\circ}$ for $\bullet, \circ \in \{+,-\}$.

Research Objectives

A 2018 paper by Antonio Lei and Bharathwaj Palvannan [2] (we will call it L-P) proves an equality of 2nd Chern classes between certain Galois-theoretic modules. They assume an equality of divisors in $Z^1(\mathbb{Z}_p[[\tilde{\Gamma}]])$ (where $Z^n(R)$ denotes the free abelian group on the height n prime ideals of R) between $\left(\text{Sel}^{\text{Gr}}(\mathbb{Q}, D_{\rho_{4,2}})^\vee\right)$ and $\theta_{4,2}^{\text{Gr}}$, as well as between $\text{Sel}^{\bullet\circ}(\mathbb{Q}, D_{\rho_{4,2}})^\vee$ and $\theta_{4,2}^{\bullet\circ}$, where $\bullet, \circ \in \{+,-\}$. Their main theorem treats every possible pair except for $\{\theta_{4,2}^{++}, \theta_{4,2}^{--}\}$ and $\{\theta_{4,2}^{+-}, \theta_{4,2}^{-+}\}$, since in these cases a key auxiliary module has rank two intead of rank one; in addition, the case of $\{++, --\}$ fails to meet a certain coprimality condition.

A 2015 paper by Greenberg et al. [3] extended results for torsion Iwasawa modules to the case of Iwasawa modules supported in codimension m. In analogy to L-P, the authors only prove their result for rank one modules. To treat the case of higher rank, they later published a paper in which they took the top exterior powers to obtain rank one modules, for which the earlier methods could be adapted..

My research this summer with Francesc Castella consisted of trying to extend Lei and Palvannan's results to the two cases in which the auxiliary module has rank two. My research objectives were to first identify the exact points in the paper's arguments where the rank two and coprimality conditions are crucially needed, and to then attempt to adapt their arguments for the exceptional cases following the analogy laid out by Greenberg et al. in taking top exterior powers. The rest of this poster is devoted to an (abridged) original derivation of an exact sequence which could lead to the correct generalization of the earlier results.

The Key Exact Sequence

We have the maps for each height two prime ideal $Q \subset R$:

$$\bigwedge^2(\text{Loc}_I(\mathbb{Q}_p, D_{\rho_{d,n}})^\vee) \otimes_R R_Q \overset{B_I}{\to} \bigwedge^2(\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}}^{**})) \otimes_R R_Q),$$

$$\bigwedge^2(\text{Loc}_{II}(\mathbb{Q}_p, D_{\rho_{d,n}})^\vee) \otimes_R R_Q \overset{B_{II}}{\to} \bigwedge^2(\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}}^{**})) \otimes_R R_Q),$$

where B_I, B_{II} are elements of R_Q since $\text{Aut}(R_Q) = R_Q$ and all four top exterior powers above are isomorphic to R_Q. For each height one prime ideal $\mathfrak{p} \subset R$, we have that the localization $\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}}) \otimes_R R_{\mathfrak{p}}$ is reflexive. We have by the same reasoning as in the paper an equality of divisors in $Z^1(R)$:

$$\text{Div}(B_1) = \text{Div}(\theta_I), \quad \text{Div}(B_2) = \text{Div}(\theta_{II})$$

and hence the existence of units $u_I, u_{II} \in R$ such that

$$B_I = u_I\theta_I, \quad B_{II} = u_{II}\theta_{II}.$$

We hence find from a certain exact sequence in L-P along with the right exactness of the exterior algebra:

$$\text{coker}\left(\bigwedge^2(\text{Loc}_I(\mathbb{Q}_p, D_{\rho_{d,n}})) \otimes_R R_Q \to \bigwedge^2(\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}})^{**}) \otimes_R R_Q\right)$$

$$\cong \bigwedge^2 \mathcal{Z}(\mathbb{Q}, D_{\rho_{d,n}}) \otimes_R R_Q.$$

Now we have the following commutative diagram of R_Q-modules:

$$\begin{array}{ccccccc} & & \bigwedge^2((\text{Loc}_I \oplus \text{Loc}_{II})^\vee)_Q & & & & \\ & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \bigwedge^2(\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}}))_Q & \longrightarrow & \bigwedge^2(\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}})^{**})_Q & \to & \bigwedge^2(\text{coker}(i_{\mathcal{X}}))_Q \to 0 \end{array}$$

where $i_{\mathcal{X}} : \mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}}) \to \mathcal{X}(\mathbb{Q}, D_{\rho_{4,2}})^{**}$ is the natural mapping into the reflexive dual, we use the obvious shorthand notation for the two localization modules, and we use the subscript notation for the localizations at Q. Observe that because all of our modules are free upon localizing and commuting the localization with the exterior power functor, we have not only exactness on the right in the bottom sequence (which is given a priori by the right exactness of the exterior power functor) but also left exactness. Now using again exactness of the exterior power functor on free modules, we have by the definitions

$$\text{coker}\left(\bigwedge^2((\text{Loc}_I \oplus \text{Loc}_{II})^\vee)_Q \to \bigwedge^2(\mathcal{X}(\mathbb{Q}, D_{\rho_{d,n}}))_Q\right) \cong R_Q/(\theta_I, \theta_{II}).$$

Hence the snake lemma applied to the above commutative diagram gives us the short exact sequence of R_Q-modules

$$0 \to \bigwedge^2(\mathcal{Z}(\mathbb{Q}, D_{\rho_{d,n}})) \otimes_R R_Q \to R_Q/(\theta_I, \theta_{II}) \to \bigwedge^2(\text{coker}(i_{\mathcal{X}})) \otimes_R R_Q.$$

References

[1] Iwasawa, Kenkichi. "On Γ-extensions of Algebraic Number Fields."*Bulletin of the American Mathematical Society*. 1959.

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[3] Bleher, F.M; Chinburg, T; Greenberg, R; Kakde, M; Pappas, G; Sharifi, R; Taylor, M.J. "Higher Chern Classes in Iwasawa Theory."*American Journal of Mathematics*. 2015.

[4] Greenberg et al. "Exterior Powers in Iwasawa Theory."arXiv preprint. 2019.

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