One Step Beyond Linear Algebra

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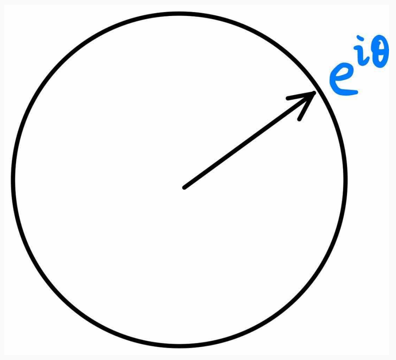
Does there exist invariant **symplectic** and **contact** forms on *almost Abelian* Lie groups?

What is. . .

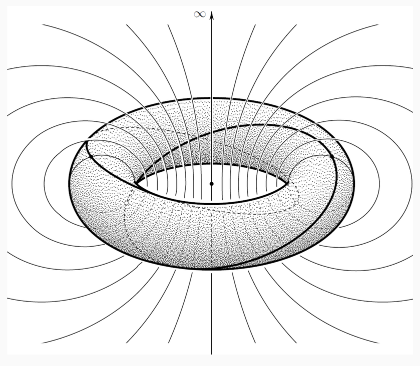
. . . a Lie group?

It is a space where the points can be multiplied with each other, satisfying nice multiplication rules, and where the product depends smoothly on the inputs.

Examples



The circle, $\mathrm{U}(1) \cong \mathrm{SO}(2)$



The 3-sphere, $\mathrm{SU}(2)$

. . . a Lie algebra?

It is a vector space with one extra operation "$[,]$", called the bracket, satisfying (i) bilinearity, (ii) antisymmetry and (iii) $[[\mathbf{x},\mathbf{y}],\mathbf{z}]+[[\mathbf{y},\mathbf{z}],\mathbf{x}]+[[\mathbf{z},\mathbf{x}],\mathbf{y}]=0$.

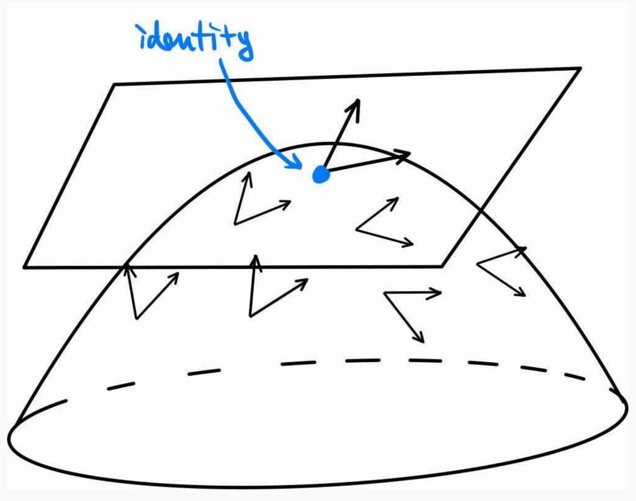
Examples

Think of the cross product in $\mathbb{R}^3$. That is, set $[\mathbf{x},\mathbf{y}] := \mathbf{x}\times\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are 3-d vectors!

. . . the Lie algebra of a Lie group?

It is the tangent space at the identity!

(. . . it is also the collection of left-invariant vector fields.)



Identity

$e\cdot g = g\cdot e = g$, $\forall g\in G$.

This e is unique.

Left translation

$L_g : h \mapsto gh$, for any $h\in G$.

This is a diffeomorphism (i.e. symmetry) of the Lie group

. . . a differential k-form?

It is a "machine" taking in k vector fields, outputting 1 function, which is antisymmetric with respect to the arguments.

$$\omega(\dots,\mathbf{v},\dots,\mathbf{w},\dots) = -\omega(\dots,\mathbf{w},\dots,\mathbf{v},\dots)$$

Example (local): determinant / oriented volume.

Exterior derivative: grad $\Rightarrow$ curl $\Rightarrow$ div, symbol: "d".

Closed: $\mathrm{d}\omega = 0$. **Non-degenerate**: $\omega(v,w)=0, \forall w \implies v = 0$.

Symplectic form: a non-degenerate, closed 2-form $\omega\in\Omega^2(M)$.

Contact form: a 1-form θ for which $\theta\wedge(\mathrm{d}\theta)^n \neq 0$ everywhere on M.

. . . left invariant?

$\omega|_g(\mathrm{d}(L_g)_e(v_1),\dots,\mathrm{d}(L_g)_e(v_k)) = \omega|_e(v_1,\dots,v_k), \quad \forall g\in G.$

$\Rightarrow \omega(\text{left-inv. } X_1,\dots,X_k) = $ const.

How about. . . . . . almost Abelian?

An **almost Abelian Lie algebra** is a non-Abelian Lie Algebra $\mathfrak{g}$ that has a co-dimension 1 Abelian ideal $\mathfrak{h}$, i.e. $\dim\mathfrak{h} = \dim\mathfrak{g} - 1$.

An **almost Abelian Lie group** is a Lie group whose Lie algebra is almost Abelian.

Abelian: A Lie algebra is abelian when $[\cdot,\cdot]\equiv 0$.

Ideal: subspace $\mathfrak{h}\subset\mathfrak{g}$ so that $[\mathfrak{h},\mathfrak{g}]\subset\mathfrak{h}$.

Examples

$$\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}$$

The Heisenberg Group

$$\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix}$$

The Heisenberg Algebra

Explaining the title

Let $\mathfrak{g}$ be an almost Abelian Lie algebra, $\mathfrak{h}\subset\mathfrak{g}$ the abelian ideal.

$$\implies \mathfrak{g} = \mathrm{Span}\{e_0\}\oplus\mathfrak{h},$$

where $[e_0, X]\in\mathfrak{h}$ for $X\in\mathfrak{h}$.

Consequence

almost abelian Lie algebra

$\Updownarrow$

vector space $\mathfrak{h}$ with specified operator $\mathrm{ad}_{e_0} : X \mapsto [e_0, X]$

Known Proposition

Operators $T', T : \mathfrak{h}\to\mathfrak{h}$ give isomorphic almost abelian Lie algebra iff

$$\lambda T' = \phi T\phi^{-1} \text{ for some } \lambda\neq 0 \text{ and invertible } \phi.$$

Consequence: $\mathfrak{g}\leftrightarrow(\mathfrak{h}=\mathbb{R}^d$ with $T=$ Jordan form).

Symplectic forms only live in **even dimensions**, while **contact forms**, **odd dimensions**.

Important Fact

Results

Proposition 1

In dimension $d=$ even, a left-invariant symplectic form exists on an almost Abelian Lie group with Lie algebra $(\mathfrak{g}, \mathrm{ad}_{e_0})$ if and only if the $(d-1)\times(d-1)$ matrix equation

$$ZJ + J^\top Z = 0$$

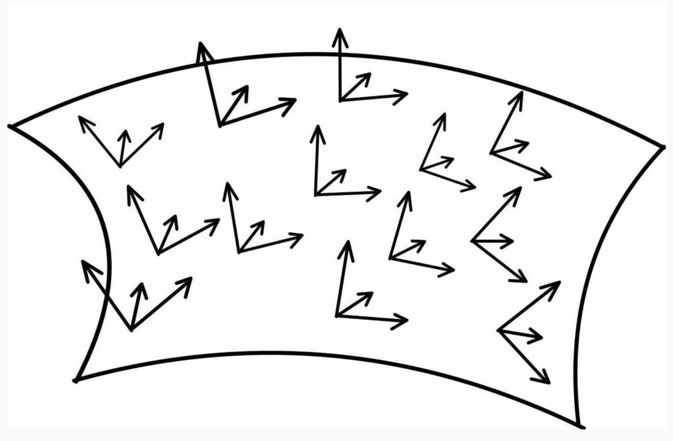
has a solution Z which is anti-symmetric and of rank $d-2$. Here J is the Jordan form of the operator ad_{e_0}.

Proposition 2

On an odd dimensional almost Abelian Lie group, therecannot exist a left-invariant contact form unless the dimension is 3.

Methods

Global Invariant Frame



Magic Formula

$$\mathrm{d}\omega(X_1,\dots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i-1}X_i\left(\omega(X_1,\dots,\widehat{X_i},\dots,X_{k+1})\right) + \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega\left([X_i,X_j],\dots,\widehat{X_i},\dots,\widehat{X_j},\dots\right),$$

Definition of Wedge Product

$$\omega\wedge\eta(V_1,\dots,V_{k+l}) := \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\mathrm{sgn}\,\sigma)\omega(V_{\sigma(1)},\dots,V_{\sigma(k)})\eta\left(V_{\sigma(k+1)},\dots,V_{\sigma(k+l)}\right),$$

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