A Section of Sheaf Cohomology

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Sheaves: The Basics

Definition 1. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on X is given by an abelian group $\mathcal{F}(U)$ for every open subset U of X so that for every pair of open sets $V \subset U$, there is a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ called a restriction. If $\sigma \in \mathcal{F}(U)$, it is often called a *section* at U and $\rho_{UV}(\sigma)$ is often denoted σ_V. The restrictions are then subject to the following requirement. If $W \subseteq V \subseteq U$ are open subsets, then

$$\rho_{VW} \circ \rho_{UV} = \rho_{UW}.$$

Definition 2. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves over a space X, then a *morphism of presheaves* $\phi : \mathcal{F} \to \mathcal{G}$ is given by a homomorphism $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U so that $\phi_U(\sigma)|_V = \phi_V(\sigma|_V)$ for all $V \subseteq U$ open subsets of X.

Definition 3. Let $\mathcal{F}$ be a presheaf. Then, it is called a *sheaf* if it satisfies the following additional conditions

i. *Locality*: If $\mathcal{U}$ is an open cover of $U \subseteq X$ and $s, t \in \mathcal{F}(U)$ so that $s|_{U_i} = t|_{U_i}$ for all $U_i \in \mathcal{U}$, then $s = t$.
ii. *Gluing*: If $s_i \in U_i$ for all $U_i \in \mathcal{U}$ so that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $U_j \in \mathcal{U}$, then there is some $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$ for all $U_i \in \mathcal{U}$.

Theorem 1. Let $\mathcal{F}$ be a presheaf over X. Then, there exists a unique sheaf $\mathcal{F}_f$ and a morphism $\phi : \mathcal{F} \to \mathcal{F}_f$ satisfying the following universal property. If $\psi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves where $\mathcal{G}$ is a sheaf, then there is a unique morphism $\tilde{\psi} : \mathcal{F}_f \to \mathcal{G}$ so that

$$\begin{array}{ccc} & & \mathcal{F}_f \\ & \overset{\phi}{\nearrow} & \downarrow \tilde{\psi} \\ \mathcal{F} & \xrightarrow{\psi} & \mathcal{G} \end{array}$$

commutes. The sheaf $\mathcal{F}_f$ is often called the *sheafification* of $\mathcal{F}$.

Example. Suppose G is an abelian group and we let $\mathcal{F}$ assign G to every open set U of X, then this assignment defines a presheaf of abelian groups over X but not a sheaf. The sheafification of this presheaf is called the *sheaf of constant stalk* G and is often defined to be the sheaf which assigns open sets U to the group of locally constant functions $f : U \to G$.

Note. Although the above definition concerned abelian groups, sheaves can be defined in more general and specific ways. For example, sheaves of sets can be defined as can sheaves of modules over a sheaf of rings. The most relevant sheaves to this poster will be the sheaf of constant stalk $\mathbb{R}$, Ω^p, the sheaf of differential $p-$forms, $\mathcal{E}$, the sheaf of holomorphic sections of a vector bundle E, and $A^{0,q}$, the sheaf of holomorphic sections of $E \otimes \Omega^{0,q}(M)$, where $\Omega^{0,q}(M)$ is the vector bundle of alternating forms of type $(0, q)$.

Resolutions

Let $\mathcal{F}$ and $\mathcal{G}$ be a sheaves over X and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

Theorem 2 The assignment

$$U \mapsto \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

defines a sheaf over X, called $\ker(\phi)$. However, the presheaf defined by

$$U \mapsto \mathrm{Im}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

does not necessarily define a sheaf. However, it is true that this presheaf is $\mathcal{G}$ if and only if ϕ is an epimorphism. We define $\mathrm{im}(\phi)$ to be the sheafification of this presheaf.

Definition 4. A long sequence of sheaves

$$0 \to \mathcal{F}_0 \xrightarrow{d_1} \mathcal{F}_1 \xrightarrow{d_2} \cdots$$

is called a *resolution* of $\mathcal{F}$ if there is an injective morphism $d_0 : \mathcal{F} \to \mathcal{F}_0$ so that $d_0(\mathcal{F}) = \ker(d_1)$ and $\mathrm{im}(d_i) = \ker(d_{i+1})$ for all i.

Theorem 3. The following are true by the Poincare lemma.

i. The sequence of sheaves of differential forms

$$0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^n \to 0$$

is a resolution of the sheaf of constant stalk $\mathbb{R}$, called the De-Rham resolution.

ii. The sequence of sheaves

$$0 \to A^{0,0} \to A^{0,1} \to \ldots A^{0,n} \to 0$$

is a resution of the sheaf $\mathcal{E}$, called the Dolbeault resultion.

Čech Cohomology

Let $\mathcal{F}$ be a sheaf over X and let $\mathcal{U} = \{U_i\}$ be a countable open conver of X indexed by $\mathbb{N}$. If $I \subseteq \mathbb{N}$, define U_I to be

$$U_I = \bigcap_{i \in I} U_i.$$

We then define the group

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_{|I|=k+1} \mathcal{F}(U_I)$$

and a function $\delta_k : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta(\sigma))_J = \sum_i (-1)^i \sigma_{J \setminus \{j_i\}} |_{U_J}$$

where $J = \{j_1, \ldots, j_{k+1}\}$ is an index so that $j_i < j_{i+1}$ for all i. We then find that $\delta_k \circ \delta_{k+1} = 0$ so these groups define a complex. Thus, we define

$$\check{H}^q(\mathcal{U}, \mathcal{F}) = \frac{\ker(\delta_q)}{\mathrm{im}(\delta_{q-1})}$$

to be the *Čech Cohomology* of $\mathcal{F}$ given this cover.

The Functor Γ

Definition 5: Let $\mathcal{F}$ be a sheaf over X. We define the *global section functor* Γ to be the functor which assigns a sheaf $\mathcal{F}$ to the object $\mathcal{F}(X)$. We note that Γ is a left exact functor

Definition 6 Let $\mathcal{F}$ be a sheaf over X and consider a resolution

$$0 \to \mathcal{F} \xrightarrow{j} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \cdots$$

where each I_i is an injective object in the category of sheaves. Then, applying Γ to this sequence gives the sequence

$$0 \to \Gamma(\mathcal{F}) \xrightarrow{\Gamma(j)} \Gamma(I_0) \xrightarrow{\Gamma(d_0)} \Gamma(I_1) \xrightarrow{\Gamma(d_1)} \Gamma(I_2) \xrightarrow{\Gamma(d_2)} \cdots$$

which remains a complex by the left exactness of Γ. We then define the qth *Sheaf Cohomology Group* to be the quotient

$$H^q(X, \mathcal{F}) = \frac{\ker(\Gamma(d_i))}{\mathrm{im}(\Gamma(d_{i-1}))}.$$

Note. Resolutions of injective objects are unique up to homotopy equivalence. Hence, sheaf cohomology groups are unique up to isomorphism.

Comparison Theorems

Theorem 5. The following isomorphisms of groups hold for real and complex manifolds respectively if $H^q_{DR}(M)$ and $H^{0,q}(M, E)$ are the De-Rham and Dolbeault cohomology groups.

i. $H^q_{DR}(M) \cong H^q(M, \mathbb{R})$
ii. $H^{0,q}(M, E) \cong H^q(M, \mathcal{E})$

Theorem 6. If $\mathcal{F}$ is a sheaf and $\mathcal{U} = \{U_i\}$ is an open cover of X so that $H^q(U_I, \mathcal{F}) = 0$ for all $q > 0$, then

$$H^q(X, \mathcal{F}) = \check{H}^q(\mathcal{U}, \mathcal{F}).$$

Theorem 7. Let X be a locally contractible space and let $H^q_{\mathrm{sing}}(X, R)$ be the singular cohomology group with coefficients in a commutative ring R. Then, if we consider $H^q(X, R)$ the qth cohomology group of the sheaf of constant stalk R, we have that

$$H^q_{\mathrm{sing}}(X, R) \cong H^q(X, R)$$

for any $q > 0$.

Example. Let M be a smooth manifold and consider the above theorem with $R = \mathbb{R}$. Then, by theorems 5 and 7

$$H^q_{\mathrm{sing}}(M, \mathbb{R}) \cong H^q(M, \mathbb{R}) \cong H^q_{DR}(M).$$

This result is a classical result in Differential Geometry called the *De-Rham Theorem*.